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On mean-field approximation of particle systems with annihilation and spikes

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On a filtered probability space let us consider the following interactions of $N(\geq 2)$ Brownian particles each of which diffuses on the nonnegative half line $\mathbb{R}_+$ and is attracted towards the average position of all the particles. When a particle i attains the boundary 0, it is annihilated (default) and a new particle (also called i) spikes immediately in the middle of particles. More precisely, let us denote by $X_t := (X_t^1, \ldots, X_t^N)$ the positions of these particles, where $X_t^i(\geq 0)$ is the position of particle i at time $t \geq 0$ for $i = 1, \ldots, N$. With the average $\overline{X}_t := (X_t^1 + \cdots + X_t^N) / N$ the dynamics of the system is determined by

$$X_t^i = X_0^i + \int_0^t b(X_s^i, \overline{X}_s)\mathrm{d}s + W_t^i + \int_0^t \overline{X}_{s-}\Big(\mathrm{d}M_s^i - \frac{1}{N}\sum_{j\neq i}\mathrm{d}M_s^j\Big); \quad t \geq 0,$$
$$M_t^i := \sum_{k=1}^{\infty} \mathbf{1}_{\{\tau_k^i \leq t\}}, \quad \tau_k^i := \inf\Big\{s > \tau_{k-1}^i : X_{s-}^i - \frac{\overline{X}_{s-}}{N}\sum_{j\neq i}(M_s^j - M_{s-}^j) \leq 0\Big\}, \tag{1}$$

for $i = 1, \ldots, N$, $k \in \mathbb{N}$, where $W_t := (W_t^1, \ldots, W_t^N)$, $t \geq 0$ is an N-dimensional Brownian motion, M_t^i is the cumulative number of defaults by time $t \geq 0$, τ_k^i is the k-th default time with $\tau_0^i = 0$ of particle i. Here we assume that $b : \mathbb{R}_+^2 \to \mathbb{R}$ is (globally) Lipschitz continuous, i.e., there exists a constant $\kappa > 0$ such that

$$|b(x_1, m_1) - b(x_2, m_2)| \leq \kappa\big(|x_1 - x_2| + |m_1 - m_2|\big) \tag{2}$$

for all $x_1, x_2, m_1, m_2 \in \mathbb{R}_+$, and we also impose the condition

$$\sum_{i=1}^{N} b(x^i, \overline{x}) \equiv 0 \tag{3}$$

for every $x := (x^1, \ldots, x^N) \in \mathbb{R}_+^N$ and $\overline{x} := (x^1 + \cdots + x^N) / N$ on the drift function $b(\cdot, \cdot)$.

Given a standard Brownian motion $W.$ we shall consider a system $X. := (X_\cdot^1, \ldots, X_\cdot^N)$, $M. := (M_\cdot^1, \ldots, M_\cdot^N))$ described by (1) with (2)-(3) on a filtered probability space $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$ with filtration $\mathbb{F} := (\mathcal{F}_t, t \geq 0)$. In particular, we are concerned with (1) that there might be multiple defaults at the same time with positive probability, i.e.,

$$\mathbb{P}\big(\exists (i,j)\ \exists t \in [0, \infty) \text{ such that } X_t^i = X_t^j = 0\big) > 0.$$

We shall construct a solution to (1) with a specific boundary behavior of defaults until the time $\overline{\tau}_0 := \inf\{s > 0 : \max_{1\leq i\leq N} X_s^i = 0\}$. Let us define the following map $\Phi(x) := (\Phi^1(x), \ldots, \Phi^N(x)) : [0, \infty)^N \mapsto [0, \infty)^N$ and set-valued function $\Gamma : \mathbb{R}_+^N \to \{1, \ldots, N\}$ defined by $\Gamma_0(x) := \{i \in \{1, \ldots, N\} : x^i = 0\}$,

$$\Gamma_{k+1}(x) := \Big\{i \in \{1, \ldots, N\} \setminus \bigcup_{\ell=1}^{k} \Gamma_\ell(x) : x^i - \frac{\overline{x}}{N} \cdot \Big|\bigcup_{\ell=1}^{k} \Gamma_\ell(x)\Big| \leq 0\Big\}; \quad k = 0, 1, 2, \ldots, N-3$$

$$\Gamma(x) := \bigcup_{k=0}^{N-2} \Gamma_k(x), \quad \Phi^i(x) := x^i + \overline{x}\Big(\Big(1+\frac{1}{N}\Big)\cdot \mathbf{1}_{\{i\in\Gamma(x)\}} - \frac{1}{N}\cdot|\Gamma(x)|\Big) \tag{4}$$

for $x = (x^1,\dots,x^N) \in \mathbb{R}_+^N$, $i = 1,\dots,N$ with $\overline{x} := (x^1+\cdots+x_N)/N \ge 0$. Note that $\Phi([0,\infty)^N \setminus \{\mathbf{0}\}) \subseteq [0,\infty)^N \setminus \{\mathbf{0}\}$ and $\Phi(\mathbf{0}) = \mathbf{0} = (0,\dots,0)$.

Lemma 1 ([3]). *Given a standard Brownian motion $W.$ and the initial configuration $X_0 \in (0,\infty)^N$ one can construct the process $(X., M.)$ which is the unique, strong solution to (1) with (2), (3) on $[0,\overline{\tau}_0]$, such that if there is a default, i.e., $|\Gamma(X_{t-})| \ge 1$ at time t, then the post-default behavior is determined by the process with $X_t^i = \Phi^i(X_{t-})$ for $i = 1,\dots,N$.*

Now let us discuss the system (1) with (2)-(3) as a mean-field approximation for nonlinear equation of McKean-Vlasov type. For the sake of concreteness, let us assume $b(x,m) = -a(x-m)$, $x,m \in [0,\infty)$ for some $a > 0$. By the theory of propagation of chaos (e.g., Tanaka (1984), Shiga & Tanaka (1985) and Sznitman (1991)) as $N \to \infty$, the dynamics of the finite-dimensional marginal distribution of limiting representative process is expressed by

$$\mathcal{X}_t = \mathcal{X}_0 - a\int_0^t (\mathcal{X}_s - \mathbb{E}[\mathcal{X}_t])\mathrm{d}s + W_t + \int_0^t \mathbb{E}[\mathcal{X}_{s-}]\mathrm{d}(\mathcal{M}_s - \mathbb{E}[\mathcal{M}_s]); \quad t \ge 0, \tag{5}$$

where $W.$ is the standard Brownian motion, $\mathcal{M}_t := \sum_{k=1}^{\infty} \mathbf{1}_{\{\tau^k \le t\}}$, $\tau^k := \inf\{s > \tau^{k-1} : \mathcal{X}_{t-} \le 0\}$, $k \ge 1$, $\tau^0 = 0$. Then taking expectations of both sides of (5), we obtain $\mathbb{E}[\mathcal{X}_t] = \mathbb{E}[\mathcal{X}_0]$, $t \ge 0$. When $\mathcal{X}_0 = x_0$ a.s. for some $x_0 > 0$, substituting this back into (5), we obtain

$$\mathcal{X}_t = \mathcal{X}_0 - a\int_0^t (\mathcal{X}_s - \mathcal{X}_0)\mathrm{d}s + W_t + \mathcal{X}_0(\mathcal{M}_t - \mathbb{E}[\mathcal{M}_t]); \quad t \ge 0.$$

Transforming the state space from $[0,\infty)$ to $(-\infty,1]$ by $\widehat{\mathcal{X}}_t := (x_0 - \mathcal{X}_t)/x_0$, we see

$$\widehat{\mathcal{X}}_t = -\int_0^t a\widehat{\mathcal{X}}_s\mathrm{d}s + \widehat{W}_t - \widehat{\mathcal{M}}_t + \mathbb{E}[\widehat{\mathcal{M}}_t]; \quad t \ge 0, \tag{6}$$

where we denote $\widehat{W}. = W./x_0$, $\widehat{\mathcal{M}}. = \mathcal{M}.$.

This transformed process $\widehat{\mathcal{X}}.$ is similar to the nonlinear McKean-Vlasov-type stochastic differential equation

$$\widetilde{\mathcal{X}}_t = \widetilde{\mathcal{X}}_0 + \int_0^t \mathrm{b}(\widetilde{\mathcal{X}}_s)\mathrm{d}s + \widetilde{W}_t - \widetilde{\mathcal{M}}_t + \alpha\mathbb{E}[\widetilde{\mathcal{M}}_t]; \quad t \ge 0, \tag{7}$$

studied by Delarue, Inglis, Rubenthaler & Tanré (2015 a,b). Here $\widetilde{\mathcal{X}}_0 < 1$, $\alpha \in (0,1)$, $\mathrm{b} : (-\infty,1] \to \mathbb{R}$ is assumed to be Lipschitz continuous with at most linear growth. $\widetilde{W}.$ is the standard Brownian motion, $\widetilde{\mathcal{M}}. = \sum_{k=1}^{\infty} \mathbf{1}_{\{\widetilde{\tau}^k \le \cdot\}}$ with $\widetilde{\tau}^k := \inf\{s > \tau^{k-1} : \widetilde{\mathcal{X}}_{s-} \ge 1\}$, $k \ge 1$, $\widetilde{\tau}^0 = 0$. When we specify $\widetilde{\mathcal{X}}_0 = 0$, $\mathrm{b}(x) = -ax$, $x \in \mathbb{R}_+$, and $\alpha = 1$, the solution $(\widehat{\mathcal{X}}., \widehat{\mathcal{M}}.)$ to (7) reduces to the solution $(\widetilde{\mathcal{X}}., \widetilde{\mathcal{M}}.)$ to (6), however, the previous study of (7) does not guarantee the uniqueness of solution to (7) in the case $\alpha = 1$.

Proposition 1 ([3]). *Assume $\mathbb{E}[\mathcal{X}_0] \ge 1$ and $b(x,m) = -a(x-m)$, $x,m \in [0,\infty)$ for some $a > 0$. There exists a unique strong solution to (5) on $[0,T]$. Moreover, for every $T > 0$, there exists a constant c_T such that every solution to (5) satisfies $(\mathrm{d}/\mathrm{d}t)\mathbb{E}[\mathcal{M}_t] \le c_T$ for $0 \le t \le T$.*

The proof is based on a fixed point argument. For example, when $a = 0$, we may reformulate the solution $(\widehat{\mathcal{X}}_\cdot, \widehat{\mathcal{M}}_\cdot)$ in (6) as

$$\widehat{\mathcal{Z}}_t = \widehat{\mathcal{X}}_t + \widehat{\mathcal{M}}_t = \widehat{W}_t + \mathbb{E}[\widehat{\mathcal{M}}_t], \quad \widehat{M}_t = \lfloor \sup_{0 \le s \le t} (\widehat{\mathcal{Z}}_s)^+ \rfloor; \quad t \ge 0, \tag{8}$$

where $\lfloor x \rfloor$ is the integer part. Given a candidate solution e_t for $\mathbb{E}[\widehat{\mathcal{M}}_t]$, $t \ge 0$, we shall consider

$$\widehat{Z}^e_t := \widehat{W}_t + e_t, \quad \widehat{\mathcal{M}}^e_t := \lfloor \sup_{0 \le s \le t} (\widehat{Z}^e_s)^+ \rfloor; \quad t \ge 0, \tag{9}$$

where the superscripts e of $\widehat{Z}^e_\cdot$ and $\widehat{\mathcal{M}}^e_\cdot$ represent the dependence on $e_\cdot$. Then uniqueness of the solution to (6) is reduced to uniqueness of the fixed point $e^*_\cdot = \mathfrak{M}_\cdot(e^*)$ of the map $\mathfrak{M} : C(\mathbb{R}_+, \mathbb{R}_+) \to C(\mathbb{R}_+, \mathbb{R}_+)$ defined by

$$\mathfrak{M}_t(e) := \mathbb{E}\big[\lfloor \sup_{0 \le s \le t} (\widehat{Z}^e_s)^+ \rfloor\big] = \mathbb{E}[\widehat{\mathcal{M}}^e_t]; \quad t \ge 0. \tag{10}$$

To solve the equation (10) let us define recursively $e^{(0)}_\cdot \equiv 0$, $e^{(n+1)}_\cdot := \mathfrak{M}_\cdot(e^{(n)})$ for $n \in \mathbb{N}_0$. Then one can verify $e^{(n)}_\cdot \le e^{(n+1)}_\cdot$ for $n \in \mathbb{N}_0$. Let us also define

$$\mathcal{L} := \{e \in C^1([0,\infty)) : \dot{e}_\cdot \ge 0, e_0 = 0, e_t \le \ell(t) := t / x_0, t \ge 0\}.$$

Then one can show that if $e^{(n)} \in \mathcal{L}$, then $e^{(n+1)} \in \mathcal{L}$. (For example, if $a = 0$ and $x_0 \ge 1$, then $\widehat{Z}_\cdot = \widehat{W}_\cdot + e_\cdot = (W_\cdot / x_0) + e_\cdot$ for every $e \in \mathcal{L}$, and hence by an application of the renewal theory

$$\mathfrak{M}_t(e) = \sum_{k=1}^{\infty} \mathbb{P}(\sup_{0 \le s \le t} (\widehat{W}_s + e_s)^+ \ge k) \le \sum_{k=1}^{\infty} \mathbb{P}(\sup_{0 \le s \le t} (W_s + s)^+ \ge k x_0) \le \frac{t}{x_0}$$

for $t \ge 0$.) By utilizing this monotone property of the map $\mathfrak{M}_t$ and the first passage time distribution for diffusions, we verify the contraction property and then find a unique fixed point in the class of continuously differentiable, nonnegative functions bounded by a linear line with slope $1 / x_0$. Note that in some numerical evaluation we observe the slow convergence of PICARD iteration even for the case $x_0 < 1$.

For the stationary distribution of the solution $\mathcal{X}_\cdot$ to (5) we have the following proposition.

Proposition 2 ([3]). *When $a > 0$, the stationary distribution of*

$$\mathcal{X}_t = \mathcal{X}_0 - a \int_0^t (\mathcal{X}_s - x_0) \mathrm{d}s + W_t + x_0(\mathcal{M}_t - \mathbb{E}[\mathcal{M}_t]); \quad t \ge 0$$

has the density

$$\mathfrak{p}_a(x) := 2c_0 \Big(\int_0^{x \wedge x_0} e^{ay^2 + 2x_0(c_0 + a)y} \mathrm{d}y \Big) e^{-ax^2 - 2x_0(c_0+a)x}; \quad x \ge 0,$$

where $c_0 := \lim_{t \to \infty} \mathrm{d}\mathbb{E}[\mathcal{M}_t] / \mathrm{d}t$ is a unique solution to

$$\frac{c_0}{a} \int_{\sqrt{2/a}c_0 - \sqrt{2a}x_0}^{\sqrt{2/a}c_0 x_0} e^{x^2/2} \Big(\int_x^{\infty} e^{-y^2/2} \mathrm{d}y \Big) \mathrm{d}x = 1.$$

When $a = 0$, we have $c_0 = 1/x_0^2$ and $\mathfrak{p}_0(x) = (1 - e^{-2x/x_0})/x_0$ if $0 < x < x_0$ and $\mathfrak{p}_0(x) = e^{-2x/x_0}(e^2 - 1)/x_0$ if $x > x_0$.

It follows from Proposition 1 that the propagation-of-chaos result holds for the reformulated solution $(\mathcal{Z}_{\cdot}, \mathcal{M}_{\cdot})$ from the original $X_{\cdot}$ in (1). Thus we have the following.

Proposition 3 ([3]). *Let us assume that X_0^i, $i \in \mathbb{N}$ are independently, identically distributed with a finite mean. Under the same assumption as in Proposition 1, for every $k \ge 1$, $\ell \ge 1$, $t_1, \dots t_\ell$, as $N \to \infty$ the vector $(X_{t_j}^i, M_{t_j}^i)$, $1 \le i \le k$, $1 \le j \le \ell$ defined from (1) converges towards the finite dimensional marginals at times $t_1, \dots, t_\ell$ of k independent copies of $(\mathcal{X}_{\cdot}, \mathcal{M}_{\cdot})$ in (5).*

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Bibliography

[1] F. DELARUE, J. INGLIS, S. RUBENTHALER, AND E. TANRÉ. (2015a) Global solvability of a networked integrate-and-fire model of Mckean–Vlasov type, *Ann. Appl. Probab.*, 25, pp. 2096–2133.

[2] F. DELARUE, J. INGLIS, S. RUBENTHALER, AND E. TANRÉ. (2015b) Particle systems with a singular mean-field self-excitation. Application to neuronal networks, *Stochastic Process. Appl.*, 125, pp. 2451–2492.

[3] R. ELIE, T. ICHIBA, AND M. LAURIERE. (2017) Large Banking Systems with Default and Recovery, *Preprint*.

[4] T. SHIGA & H. TANAKA. (1985) Central limit theorem for a system of Markovian particles with mean field interactions, *Probab. Theory Related Fields* 69 (3) (1985) 439–459.

[5] A.-S. SZNITMAN. (1991) Topics in propagation of chaos, in: *École d'Été de Probabilités de Saint-Flour XIX* –1989, Springer, pp. 165–251.

[6] H. TANAKA. (1984) Limit theorems for certain diffusion processes with interaction, in *Stochastic Analysis* (Katata/Kyoto, 1982), North-Holland Math. Library 32, 469–488.

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